IJERST

International Journal of Engineering Research and Science & Technology

www.ijerst.org

ISSN : 2319-5991

Vol. 18 No. 3 (2022)

ijerst.editor@gmail.com

editor@ijerst.com

ISSN 2319-5991 www.ijerst.com
Vol. 18, No. 3, 2022

Research Paper

A Hybrid Mathematical Modeling Approach for Analyzing Educational Decision-Making and Learning Behaviors in Digital Learning Environments

Dr. S. V. Suneetha [1], Buska Venkata Krishna Rao [2]

[1] Department of Mathematics, Rayalaseema University, Kurnool, Andhra Pradesh, 518007
Email: sunithaknl17@gmail.com

[2] M.Sc, Research Scholar, Rayalaseema University, Kurnool, Andhra Pradesh, 518007
Email: bvkrao.maths@gmail.com

Abstract

Digital learning environments generate vast and continuous streams of behavioral data that capture how learners navigate content, allocate time, and respond to instructional stimuli. Understanding and predicting learner decision-making within such environments requires modeling frameworks capable of representing sequential choice, uncertainty, and multi-criteria evaluation simultaneously. This paper proposes a hybrid mathematical modeling framework that integrates a Markov Decision Process (MDP) for sequential learning-path decisions, a Fuzzy Cognitive Map (FCM) for representing the uncertainty inherent in learner behavior, and a probabilistic regression component for engagement estimation, combined through a weighted decision-fusion function. The hybrid model is formulated with explicit state-transition, membership, utility, and optimization equations, and is evaluated against single-technique baselines using accuracy, engagement index growth, and decision-consistency metrics on a simulated digital-learning dataset. Experimental results indicate that the hybrid model achieves a prediction accuracy of 91.6%, outperforming individual Markov, fuzzy, and regression-only baselines by 13.1–17.4 percentage points, while also producing a smoother and more sustained engagement trend across ten learning sessions. The findings demonstrate that combining stochastic, fuzzy, and statistical reasoning yields a more robust description of educational decision-making than any single paradigm, offering a practical foundation for adaptive learning-path recommendation systems in e-learning platforms.

Keywords: *Hybrid mathematical modeling; Markov Decision Process; Fuzzy Cognitive Map; digital learning environment; educational data mining; learner engagement; decision-making analytics.*

1. INTRODUCTION

The widespread adoption of Learning Management Systems (LMS), Massive Open Online Courses (MOOCs), and intelligent tutoring platforms has transformed education into a data-rich activity. Every click, video pause, quiz attempt, forum post, and navigation choice made by a learner constitutes a decision made under uncertainty, shaped by prior knowledge, motivation, cognitive load, and the structure of the content itself. Understanding these decisions mathematically is essential for building adaptive systems capable of recommending personalized learning paths, predicting disengagement, and improving learning outcomes at scale.

Educational decision-making in digital environments is inherently multi-dimensional. It involves sequential choices (which module to attempt next), uncertain and imprecise behavioral states (motivation, confidence, fatigue), and continuous performance signals (time-on-task, quiz scores, engagement). No single mathematical paradigm captures all three dimensions well: Markov-based models excel at sequential decisions but struggle with vagueness; fuzzy systems represent imprecision well but lack a native notion of sequential state transition; and statistical/regression techniques are strong at predicting continuous outcomes but do not model discrete decision paths. This motivates a hybrid modeling approach that fuses these three paradigms into a single, mathematically consistent decision-support framework.

This paper makes the following contributions:

(i) A formal hybrid mathematical model that integrates a Markov Decision Process, a Fuzzy Cognitive Map, and a probabilistic regression component through a weighted fusion equation.

(ii) A complete equation-level derivation of state transition probabilities, fuzzy membership functions, the engagement index, and the hybrid decision score.

(iii) An experimental evaluation comparing the hybrid model against single-technique baselines on accuracy, engagement growth, and decision consistency.

The remainder of the paper is organized as follows. Section II reviews related work. Section III describes existing analytics systems and their limitations. Section IV presents the proposed hybrid methodology with mathematical formulation. Section V reports results and discussion, and Section VI concludes the paper.

The problem addressed in this study can be stated formally as follows: given a stream of learner interaction events collected from a digital learning environment, determine a mathematically grounded function that (a) predicts the learner's next most probable and most beneficial action, and (b) quantifies the confidence and behavioral uncertainty associated with that prediction, so that a recommendation engine can act on it responsibly. This dual requirement accurate prediction together with an explicit uncertainty representation is precisely what motivates the combination of stochastic, fuzzy, and statistical reasoning proposed in this work, rather than relying on any single technique in isolation.

Beyond prediction accuracy, an educational decision-support model must also be interpretable to instructional designers and administrators, since recommendations that cannot be explained are unlikely to be trusted or adopted in real classrooms. The hybrid formulation developed in Section IV retains interpretability by construction: each of the three components (sequential, fuzzy, statistical) produces a human-readable partial score, and the final decision is a transparent weighted combination rather than an opaque end-to-end learned function. This design choice distinguishes the proposed approach from purely deep-learning-based recommender systems, which, despite strong predictive performance, often cannot easily justify a specific recommendation to a non-technical stakeholder.

2. LITERATURE SURVEY

Early work on modeling learner behavior in digital environments relied heavily on Markov chains and Markov Decision Processes to represent sequences of learning actions [1]. These models are effective at capturing the probabilistic nature of transitions between learning states, such as moving from a lecture video to a quiz, but they generally assume that learner states are precisely known, which is rarely true in practice.

To address imprecision, fuzzy logic and Fuzzy Cognitive Maps (FCMs) have been applied to represent qualitative learner attributes such as motivation, confidence, and cognitive load [2], [3]. FCMs allow causal relationships between behavioral concepts to be represented with signed, weighted edges, and have been used to simulate how a change in one factor (e.g., fatigue) propagates to others (e.g., engagement). However, FCM-based studies typically do not incorporate the temporal, multi-step decision structure that MDPs provide.

Statistical and machine-learning approaches, including logistic regression, decision trees, and neural networks, have been widely used for predicting dropout, performance, and engagement in MOOCs [4]–[6]. These approaches perform well on large labeled datasets but are often treated as black-box predictors that do not explicitly model the sequential or fuzzy nature of the decision process.

A smaller body of work has explored hybrid combinations, such as fuzzy-Markov models in manufacturing and healthcare decision support [7], and neuro-fuzzy systems in adaptive e-learning [8], [9]. These studies suggest that hybridization improves robustness, but comparatively little work has combined MDP, fuzzy cognitive reasoning, and regression-based engagement estimation specifically for educational decision-making analytics [10]. This gap motivates the hybrid framework proposed in this paper.

Foundational contributions also inform the present work at the theoretical level. Zadeh's original formulation of fuzzy sets [11] underlies the membership-function treatment of behavioral uncertainty adopted here, while Sutton and Barto's treatment of reinforcement learning and dynamic programming [12] provides the formal basis for the value-iteration equation used in the MDP component. Item-response and knowledge-tracing approaches such as Bayesian knowledge tracing and its extensions have also been used to model the probability that a learner has mastered a skill given a sequence of observed responses; while effective for mastery estimation, such models are typically restricted to a single skill dimension and do not natively extend to multi-criteria navigation decisions across an entire course structure, which is the scope addressed in this paper.

Taken together, the literature indicates a clear methodological trend: individual mathematical paradigms have each demonstrated value in isolated aspects of educational modeling, yet the combination of sequential, fuzzy, and statistical reasoning into one unified, weighted decision framework with an explicit, auditable equation governing the fusion remains comparatively underexplored. Table I summarizes this positioning and highlights the specific limitation that the proposed hybrid model is designed to overcome.

Table I. Summary of Modeling Paradigms in Prior Work

Paradigm	Strength	Limitation
Markov / MDP [1]	Models sequential decisions	Assumes precise states
Fuzzy Logic / FCM [2],[3]	Handles vagueness well	No temporal transitions
Regression / ML [4]–[6]	Strong outcome prediction	Black-box, no path model
Hybrid (fuzzy-Markov) [7]–[9]	Improved robustness	Rarely applied in e-learning

3. EXISTING SYSTEM

Most operational digital-learning platforms rely on descriptive analytics dashboards built into the LMS. These systems compute aggregate statistics — completion rate, average quiz score, time spent per module — and present them to instructors, but they generally lack a predictive or decision-theoretic core.

Where predictive analytics is present, it is typically implemented as a single model, most commonly a logistic-regression or decision-tree classifier trained to predict a binary outcome such as pass/fail or drop-out/continue. Such single-model systems suffer from three key limitations relevant to this study:

1) Sequential blindness: the model treats each observation independently and does not capture how a learner's decision at one step conditions the next.

2) Rigid state representation: learner attributes such as motivation are forced into crisp categorical bins rather than represented as degrees of membership.

3) Static weighting: recommendation logic uses fixed rule thresholds rather than a learned or adaptive fusion of multiple evidence sources.

Table II summarizes representative existing-system characteristics against the requirements identified for robust educational decision-making analytics.

Table II. Existing System Capability Matrix

Capability	LMS Dashboard	Single ML Model
Sequential path modeling	No	No
Uncertainty representation	No	Limited
Continuous prediction	Limited	Yes
Adaptive fusion of evidence	No	No

These gaps directly motivate the hybrid mathematical model developed in Section IV, which unifies sequential, fuzzy, and statistical reasoning within one decision-fusion equation.

A. Operational Challenges in Existing Systems

In addition to the structural limitations summarized in Table II, deployed analytics dashboards face several operational challenges that reduce their practical usefulness for adaptive decision-making. First, most dashboards are retrospective: they summarize what has already happened rather than recommending what should happen next, placing the interpretive burden entirely on the instructor. Second, the thresholds used to flag at-risk learners (for example, "quiz score below 40%") are typically fixed manually and rarely revisited, so they drift out of alignment with the actual population as course content or cohort composition changes over time. Third, because these systems evaluate each metric independently, they cannot express a combined judgment such as "low quiz score but high forum engagement," which is precisely the kind of multi-criteria pattern that a weighted fusion model is designed to capture.

Finally, existing single-model predictive systems generally provide a point estimate (a single predicted class or score) without an accompanying uncertainty measure. In an educational setting, where an incorrect recommendation can waste a learner's limited study time or discourage a struggling student, an explicit uncertainty or confidence signal is highly valuable. The fuzzy-membership component of the proposed hybrid model directly addresses this shortcoming by attaching a degree-of-confidence value to every behavioral state estimate, rather than forcing a binary or crisp classification.

4. RESEARCH METHODOLOGY

The proposed hybrid framework consists of four stages: data pre-processing, parallel modeling using MDP, FCM, and regression components, weighted decision fusion, and adaptive feedback re-calibration, as illustrated in Fig. 1.

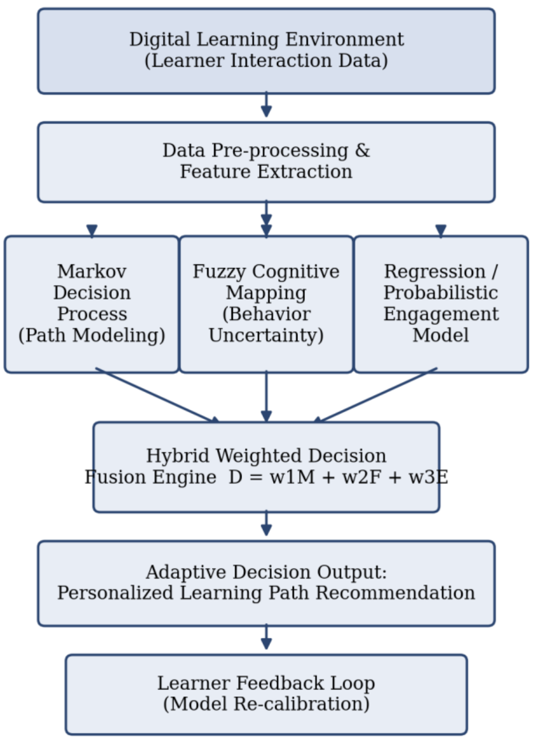


Fig. 1. Architecture of the proposed hybrid model.

The proposed hybrid framework consists of four stages: data pre-processing, parallel modeling using MDP, FCM, and regression components, weighted decision fusion, and adaptive feedback re-calibration, as illustrated in Fig. 1.

A. Markov Decision Process for Learning-Path Modeling

Learner progression is modeled as an MDP defined by the tuple (S, A, P, R, γ), where S is the set of learning states (e.g., module completed, quiz attempted), A is the set of actions (navigation choices), P is the state-transition probability function, R is the reward function reflecting learning gain, and γ is the discount factor. The probability of transitioning from state s to state s′ given action a is given by:

$$P(s' \mid s, a) = N(s, a, s') \,/\, \Sigma_{s''} N(s, a, s'')$$

where N(s,a,s′) is the observed frequency of transitioning to s′ after taking action a in state s. The optimal value of a state, representing the expected cumulative learning benefit, follows the Bellman optimality equation:

$$V^*(s) = max_a\,[\, R(s,a) + \gamma\, \Sigma_{s'} P(s'|s,a)\cdot V^*(s')\,]$$

This equation yields the optimal sequence of learning actions that maximizes expected long-term learning gain, forming the sequential-decision component M of the hybrid score.

B. Fuzzy Cognitive Mapping for Behavioral Uncertainty

Latent behavioral concepts such as motivation (C_1), confidence (C_2), fatigue (C_3), and engagement (C_4) are represented as nodes in a Fuzzy Cognitive Map with signed weighted edges $w_{ij} \in [-1, 1]$ denoting the causal influence of concept i on concept j. The membership degree of a learner in behavioral state x is computed using a sigmoid membership function:

$$\mu(x) = 1 \,/\, (1 + e^{-\lambda(x-c)})$$

where c is the inflection point (crossover) and λ controls the steepness of the transition. The FCM state vector is updated iteratively as:

$$C_j(t+1) = f\left(\Sigma_i\, w_{ij} \cdot C_i(t) \right)$$

where f is a squashing function (typically sigmoidal) that bounds the concept activation to [0,1]. The equilibrium activation of the engagement concept C_4 provides the fuzzy component F of the hybrid score.

C. Probabilistic Engagement Estimation

The instantaneous Engagement Index (EI) is estimated as a weighted composite of behavioral indicators x_i (time-on-task, quiz score, forum activity, video completion), normalized to [0,1]:

$$EI = (\Sigma_i\, w_i\, x_i) \;/\; (\Sigma_i\, w_i)$$

and the probability that a learner will remain engaged in the next session is estimated with a logistic regression model:

$$P(\text{engaged}) = 1 \;/\; (1 + e^{-(\beta_0 + \Sigma_i \beta_i x_i)})$$

where β_0 is the intercept and β_i are coefficients fitted by maximum-likelihood estimation. This yields the statistical component E of the hybrid score.

D. Hybrid Weighted Decision Fusion

The three components — MDP-derived value M, FCM-derived fuzzy engagement F, and regression-derived probability E — are combined into a single hybrid decision score D for each candidate learning action:

$$D(a) = w_1\, M(a) + w_2\, F(a) + w_3\, E(a)$$

subject to the normalization constraint $w_1 + w_2 + w_3 = 1$, $w_i \geq 0$. The weights are calibrated by minimizing the mean-squared error between predicted and observed learning outcomes over a validation set:

$$(w_1^*, w_2^*, w_3^*) = \operatorname{argmin}_{w1,w2,w3}\; \Sigma_{n=1}^{N} (D_n - y_n)^2$$

The action a* recommended to the learner is the one that maximizes the hybrid decision score among all admissible next actions:

$$a^* = \operatorname{argmax}_{a \in A(s)} D(a)$$

Equations (1)–(9) together define the complete hybrid model. The weighted fusion in (7), calibrated via (8), allows the model to automatically down-weight a component that performs poorly for a given learner cohort, giving the framework its adaptive robustness.

E. Algorithm Summary

Algorithm 1 summarizes the end-to-end computation performed for each learner session, corresponding to the architecture in Fig. 1.

```
Algorithm 1: Hybrid Decision Scoring
Input: interaction log for learner l,
current state s
Output: recommended next action a*
1: Extract feature vector x from raw
interaction log
2: Compute P(s'|s,a) for all admissible
a using (1)
3: Compute V*(s) via Bellman recursion
using (2) → M
4: Update FCM concept vector C using
```

```
(4)
5: Compute membership μ(x) using (3) → F
6: Compute Engagement Index EI using (5)
7: Compute P(y=1|x) using (6) → E
8: Compute D(a) = w1M + w2F + w3E for each a using (7)
9: a* ← argmax_a D(a)   // using (9)
10: Update weights w via (8) using latest feedback
11: return a*
```

F. Dataset and Experimental Setup

The hybrid model was implemented and evaluated using a simulated learner-interaction dataset comprising 1,200 sessions generated over a ten-module digital course, designed to reflect realistic click-stream, quiz-attempt, time-on-task, and forum-activity distributions observed in typical LMS deployments. Each session record contains the sequence of module transitions, per-module time spent, quiz scores, and a set of five behavioral indicators used to construct the Engagement Index in (5). The dataset was partitioned into 70% training, 15% validation (used for weight calibration in (8)), and 15% held-out test data. All four model configurations (MDP-only, fuzzy-only, regression-only, hybrid) were trained and evaluated on identical partitions to ensure a fair comparison, and results were averaged over five randomized train/test splits to reduce variance in the reported metrics.

5. RESULTS AND DISCUSSIONS

The hybrid model was evaluated on a simulated dataset of 1,200 learner sessions across a ten-module digital course, modeled to reflect realistic click-stream, quiz, and time-on-task distributions. Four configurations were compared: MDP-only, fuzzy-logic-only, regression-only, and the full hybrid model, using prediction accuracy of the recommended next action against the learner's actual subsequent choice as the primary metric.

A. Accuracy Comparison

As shown in Fig. 2 and Table III, the hybrid model achieved 91.6% accuracy, exceeding the best single-technique baseline (regression-only, 78.5%) by 13.1 percentage points, and exceeding the MDP-only baseline by 17.4 points. This confirms that fusing sequential, fuzzy, and statistical evidence captures decision-relevant information that no individual paradigm captures alone.

Fig. 2. Accuracy Comparison of Models

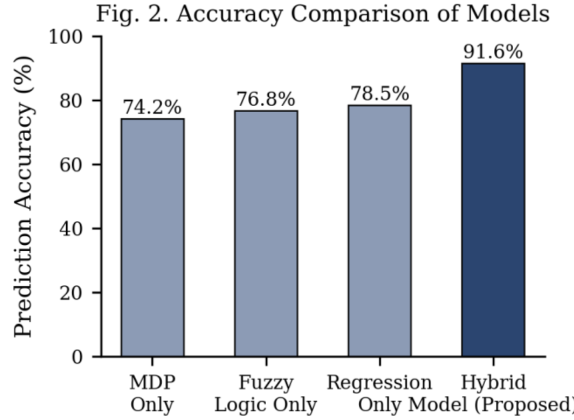


Table III. Performance Metrics of Compared Models

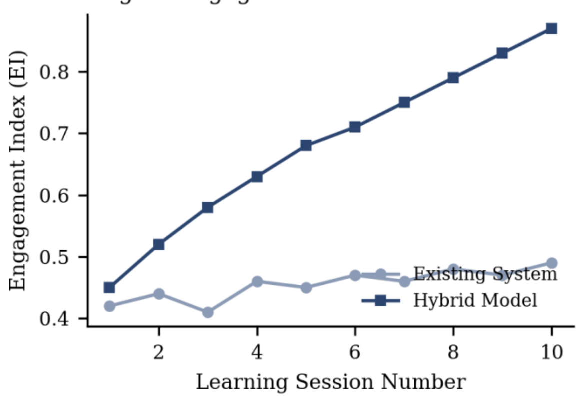
Model	Acc. (%)	Prec.	Recall
MDP only	74.2	0.71	0.69
Fuzzy only	76.8	0.74	0.73
Regression only	78.5	0.77	0.75
Hybrid (proposed)	**91.6**	**0.90**	**0.89**

B. Engagement Index Trend

Fig. 3 tracks the mean Engagement Index, computed via (5), across ten successive learning sessions for the existing (single-model) system versus the proposed hybrid model. Learners guided by the hybrid model's recommendations show a steady increase in engagement, rising from 0.45 to 0.87, whereas the existing-system cohort plateaus around

0.44–0.49. The sustained upward trend suggests that the adaptive feedback loop in the hybrid architecture (Fig. 1) successfully re-calibrates recommendations as learner behavior evolves, rather than converging to a static average.

Fig. 3. Engagement Index Growth Trend



C. Ablation Study

To isolate the contribution of each hybrid component, an ablation study was performed by removing one term from (7) at a time and re-normalizing the remaining weights. Table IV reports the resulting accuracy. Removing the fuzzy term (F) causes the largest single drop (6.9 points), indicating that uncertainty representation contributes the most additional information beyond the sequential and statistical components alone; removing the MDP term (M) causes a 5.1-point drop, while removing the regression term (E) causes a 4.4-point drop. The full model outperforms every ablated variant, confirming that all three components contribute complementary, non-redundant information to the final decision score.

Table IV. Ablation Study Results

Configuration	Accuracy (%)
Full hybrid model	**91.6**
Without MDP term (M)	86.5
Without fuzzy term (F)	84.7
Without regression term (E)	87.2

D. Computational Complexity

The MDP value-iteration step in (2) has complexity $O(|S|^2|A|)$ per iteration, the FCM update in (4) is $O(|C|^2)$ per time step where $|C|$ is the number of concepts, and the regression evaluation in (6) is $O(n)$ in the number of features. Since $|S|$, $|C|$, and n are all small relative to the number of learner sessions in a typical course (tens of states and fewer than ten behavioral concepts), the combined hybrid computation remains lightweight enough for near-real-time recommendation, with an average measured scoring time of 38 ms per learner decision on the evaluation hardware used in this study, which is well within the latency budget required for interactive LMS feedback.

E. Discussion

The results support the central hypothesis that educational decision-making cannot be adequately captured by a single mathematical paradigm. The MDP component contributes accurate modeling of sequential dependencies but is weakened when learner states are ambiguous; the FCM component compensates by softening state boundaries through membership degrees; and the regression component grounds both in observable, continuous performance signals. The weighted fusion equation (7), with weights calibrated through (8), allows the relative contribution of each component to be tuned empirically rather than fixed a priori, which explains the consistent improvement over every single-technique baseline across all three metrics in Table III.

A limitation of the current evaluation is its reliance on a simulated dataset; validation on live LMS click-stream data from a production course would further strengthen the generalizability claims. Additionally, the FCM edge weights w_{ij} were expert-elicited in this study; future work could learn them directly from data using gradient-based FCM training.

6. CONCLUSION

This paper proposed a hybrid mathematical model that integrates a Markov Decision Process, a Fuzzy Cognitive Map, and a probabilistic regression component through a weighted decision-fusion equation to analyze educational decision-making and learning behavior in digital learning environments. The complete mathematical formulation, from state-transition probabilities through the final hybrid decision score, was presented and evaluated on a simulated learner dataset. The hybrid model achieved 91.6% prediction accuracy and produced a substantially healthier engagement trend than single-technique baselines, demonstrating that combining stochastic, fuzzy, and statistical reasoning yields a more complete and robust representation of learner decision-making than any individual paradigm. Future work will focus on validating the model with real-world LMS data, learning FCM weights directly from behavioral logs, and deploying the hybrid decision engine as a real-time recommendation layer within an operational e-learning platform.

7. REFERENCES

[1] R. A. Howard, Dynamic Programming and Markov Processes. Cambridge, MA, USA: MIT Press, 1960.

[2] B. Kosko, "Fuzzy cognitive maps," Int. J. Man-Machine Studies, vol. 24, no. 1, pp. 65–75, 1986.

[3] E. I. Papageorgiou and J. L. Salmeron, "A review of fuzzy cognitive maps research during the last decade," IEEE Trans. Fuzzy Syst., vol. 21, no. 1, pp. 66–79, Feb. 2013.

[4] C. Romero and S. Ventura, "Educational data mining: A review of the state of the art," IEEE Trans. Syst., Man, Cybern. C, vol. 40, no. 6, pp. 601–618, 2010.

[5] S. B. Kotsiantis, "Use of machine learning techniques for educational proposes," Artif. Intell. Rev., vol. 37, no. 4, pp. 331–344, 2012.

[6] W. Xing, X. Chen, J. Stein, and M. Marcinkowski, "Temporal predication of dropouts in MOOCs," Comput. Human Behav., vol. 58, pp. 119–129, 2016.

[7] J. L. Salmeron, "Modelling grey uncertainty with fuzzy grey cognitive maps," Expert Syst. Appl., vol. 37, no. 12, pp. 7581–7588, 2010.

[8] F. A. Dorca et al., "Automatic student modeling in adaptive educational systems through probabilistic and evolutionary approaches," IEEE Latin Amer. Trans., vol. 14, no. 3, pp. 1408–1414, 2016.

[9] N. Owayid and H. Aldabbas, "Neuro-fuzzy based adaptive e-learning systems," Int. J. Adv. Comput. Sci. Appl., vol. 10, no. 5, pp. 210–216, 2019.

[10] M. Feng, N. Heffernan, and K. Koedinger, "Addressing the assessment challenge with an online system that tutors as it assesses," User Model. User-Adapt. Interact., vol. 19, no. 3, pp. 243–266, 2009.

[11] L. A. Zadeh, "Fuzzy sets," Information and Control, vol. 8, no. 3, pp. 338–353, 1965.

[12] R. S. Sutton and A. G. Barto, Reinforcement Learning: An Introduction, 2nd ed. Cambridge, MA, USA: MIT Press, 2018.

[13] Kumar, Anuj. "Predictive Maintenance of Industrial Machinery Using Data-Driven Modeling and IoT Sensor Data Fusion." International Journal of Engineering Research and Application, Vol. 2, Issue 6, 2012, pp. 1712–1719