

Research Paper

Analysis of Casson Nanofluid Dynamics Over a Vertical Stretching/Shrinking Surface

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Abstract

An investigation is carried out to understand the behavior of Casson fluid, specifically modeling blood infused with copper (Cu) nanoparticles, to enhance its potential in physiological and industrial applications. The focus is on flow over a shrinking/stretching sheet under mixed convection conditions. The nonlinear partial differential equations (PDEs) governing the flow are reduced to a set of nonlinear ordinary differential equations (ODEs) via similarity transformations obtained through Lie group analysis. These resulting ODEs are subsequently solved through numerical means employing the fourth-order Runge—Kutta algorithm, with validation performed against established results. Additionally, MATLAB's built-in bvp4c function is employed to solve the nonlinear ODEs. The study examines the effects of various parameters, including stretching/shrinking, volume fraction, and Casson parameters, on Nusselt number, skin friction, velocity, and thermal profiles, as illustrated through graphical representations. The results reveal that an increase in both the Casson parameter and the nanoparticle volume fraction enhances the fluid velocity. Additionally, greater values of the mixed convection parameter, Prandtl number, and Casson parameter contribute to improved heat transfer rates. The uniqueness of this study lies in the application of the Casson fluid model to practical medical scenarios, including cancer therapy via magnetic hyperthermia, as well as its prospective role in the development of advanced biomedical devices.

Keywords: Casson fluid; Stretching/shrinking Sheet; Mixed convection; MATLAB; Shooting method.

1. Introduction

Flow and heat transfer of a viscous fluid past a stretchable sheet [1] is a significant problem in fluid dynamics. The study of the flow over a stretching sheet [2] is also relevant in the field of metallurgy and chemical engineering. Stretching/shrinking sheets occurs in many

industries. Some applications are energy absorption, smart textiles, soft robotics, filtration, and separation. Ramudu et al. [3] analyzed the convective transference of heat for Casson fluid flow on a vertically extending surface. They stated that the local skin friction is directly related to the Grashof number. Using a semi-analytical method, Sohail et al. [4] calculated the flow field chaos on a nonlinear bi-directional extended sheet. Khan et al. [5] found that stretching or shrinking the sheet affects the local Nusselt number and friction coefficient. Additional studies of Casson and other nanofluid flow in different applications were reported in [6, 7, 8]. These studies have enhanced our understanding of nanofluid behavior and paved the way for better design and optimizing fluid flow for different applications.

Non-Newtonian fluids are those in which the shear stress is not linearly related to the shear rate. Such fluids are commonly found in both industrial processes and biological systems- examples include lubricants, paints, pulp suspensions, polymer-based solutions, and various biological fluids such as honey, apple sauce, and blood. The complex rheological behavior of non-Newtonian fluids makes the classical Navier—Stokes equations insufficient for accurately describing their motion, thereby posing intriguing challenges for researchers. Owing to the wide-ranging applications of non-Newtonian nanofluids, there has been significant interest in this area, with numerous studies [9] contributing valuable insights to the field of fluid mechanics.

The investigation of non-Newtonian fluid dynamics—particularly heat transfer behavior over stretching surfaces—has attracted considerable research interest due to its significance in a range of industrial and engineering fields, including polymer extrusion, electronic device cooling, metal casting, food manufacturing, and biomedical systems. Considering the diverse types of non-Newtonian models, the Casson fluid model stands out for its suitability in representing yield-stress fluids such as blood, chocolate, honey, and certain polymer-based substances. The intricate rheological behavior of Casson fluids calls for the formulation of sophisticated mathematical frameworks and numerical approaches to evaluate flow and thermal characteristics under varying boundary scenarios. Seminal contributions by Srivastava and Saxena [10] highlighted the utility of the Casson model in both biomedical and industrial contexts, setting the stage for subsequent research on its behavior over stretching surfaces. For example, Konai et al. [11] explored the influence of Stefan blowing on Casson fluid flow over a stretching boundary, enhancing the understanding of fluid-surface interactions. Furthermore, Wang et al. [12] conducted numerical simulations and applied statistical tools to analyze Casson fluid dynamics. More recent investigations [13, 14] have examined Casson nanofluids under diverse thermophysical influences, offering deeper insights into their improved heat transfer performance in stretching flow configurations.

Mixed convection describes a flow regime where both forced convection, generated by external mechanisms like fans or pumps, and natural convection, driven by thermal or concentration-induced buoyancy forces, simultaneously influence fluid motion. The balance between these two mechanisms is governed by multiple factors, including the geometry of the system, the magnitude of temperature gradients, and the physical properties of the fluid. This phenomenon is commonly encountered in real-world scenarios, such as in thermal regulation of electronic equipment, HVAC systems in buildings, industrial operations, and environmental flow systems, where mechanical and buoyancy forces act together. Accurately analyzing mixed convection flows is important in these applications due to the complexity

introduced by the interplay of both convection types. Proper modeling must incorporate con-

tributions from both mechanisms, making it more challenging than modeling purely natural or purely forced convection. Several related studies are discussed in the literature, including those in [15, 16].

Researchers are interested in the non-Newtonian fluid flow with mixed convection because of its extensive use in industry and manufacturing. As a result, the authors provide an in-depth study of how mixed convective effects on temperature impact heat and non-Newtonian Casson fluid flow. The governing PDEs (partial differential equations) are transformed into a system of ODEs (ordinary differential equations) through similarity transformations which is obtained through Lie group analysis. These resulting ODEs are then solved numerically using the fourth-order Runge—Kutta method, with the results validated against established benchmarks. The computed data are presented and interpreted through graphical analysis. The findings are crucial for optimizing heat transfer and ensuring fluid stability in various advanced engineering systems. The practical implications are broad: Casson fluids can significantly improve coolant efficiency and minimize drag in automotive engines. In energy-related technologies, these nanofluids play a key role in effective thermal regulation, particularly in applications like nuclear reactors, air conditioning systems, and heat exchangers.

2. Mathematical modelling

In fluid dynamics, analyzing fluid flow over vertically stretchable or shrinkable surfaces is vital for numerous industrial and biomedical processes, including the extrusion of polymer and metal sheets, drawing of plastic films, manufacturing of paper and glass fibers, food processing, material insulation, and geothermal energy applications. This study explores the transport behavior of a Casson nanofluid, specifically considering copper nanoparticles uniformly suspended in blood, over a vertical solid surface. The surface, immersed in a stationary fluid, is assumed to undergo stretching or shrinking along the vertical $\bar{x}$ -axis, thereby inducing a steady boundary layer flow, as illustrated in Fig. (1). Additionally, constant velocity and temperature conditions are assumed in the region outside the boundary layer. The velocity at surface and far from surface is $\bar{u} = \bar{u}_w = b\bar{x}$, and $\bar{u} = \bar{u}_e = a\bar{x}$, respectively where $a > 0$ represents the rate at which fluid flows at free stream and b is stretching/shrinking rate.

The Casson fluid model, as introduced by Casson [17], is commonly employed to describe the behavior of non-Newtonian fluids, including substances such as printing inks, blood, liquid detergents, cosmetics, fruit juices, tomato sauce, and similar materials. This model is particularly useful for describing fluids that exhibit a yield stress,

$$\tau_{ij} = 2 \begin{cases} \left[\mu + \frac{\gamma_y}{2\pi} \right] e_{ij}, & \pi > \pi_c \\ \left[\mu + \frac{\gamma_y}{2\pi_c} \right] e_{ij}, & \pi < \pi_c \end{cases}$$

Here τ_{ij} are components of the stress tensor, $\gamma_y = \frac{\mu \sqrt{2\pi}}{\beta}$ is a finite value of Casson yield



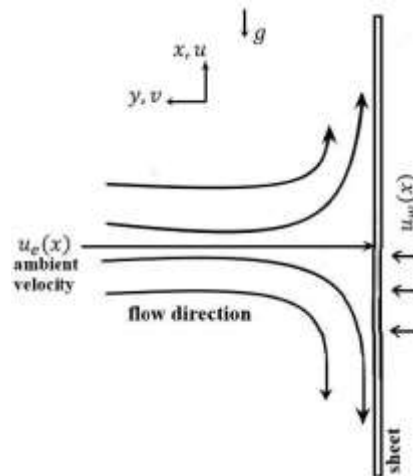


Figure 1: Flow model geometry

the coefficient for dynamic viscosity, θ represents the Casson fluid parameter, and the stress tensor τ denotes the internal stress distribution within the fluid. The quantity $\pi = e_{ij}e_{ji}$ (where e_{ij} is the $(i, j)^{th}$ component of the rate of deformation tensor) is used to characterize the fluid's internal strain and π_c represents the critical value for non-Newtonian behavior. As $\theta \rightarrow \infty$, the fluid's non-Newtonian nature diminishes, and it approaches the behavior of a Newtonian fluid. Thus, the dynamic viscosity for Casson fluid is conferred as

$$\mu^* = \mu + \frac{\gamma_c}{\sqrt{2\pi}} = \mu \left(1 + \frac{1}{\theta} \right)$$

Based on the stated assumptions and the constitutive relation, the flow dynamics are formulated through a set of non-linear coupled differential equations. Applying the boundary layer approximation principle as outlined in [18], minor variations in the flow characteristics are disregarded, leading to a two-dimensional formulation of the governing equations within the Cartesian coordinate framework. Given that the flow is considered incompressible and steady, the mass conservation law simplifies to the continuity equation in the following form:

$$\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{u}}{\partial \bar{x}} = 0, \quad (1)$$

In accordance with Boussinesq's approximation theory [19], the variation in fluid density responsible for the buoyancy force is incorporated. This leads to the formulation of the linear momentum equation as follows:

$$\bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} = \bar{u}_e \frac{d\bar{u}_e}{d\bar{x}} + \frac{\mu_{nf}}{\rho_{nf}} \left(1 + \frac{1}{\theta} \right) \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{(\rho\beta)_{nf}}{\rho_{nf}} (T - T_\infty) g, \quad (2)$$

where g represents the gravitational acceleration.

The corresponding energy equation for the present problem is expressed as follows:

$$(\rho C_p)_{nf} \bar{v} \frac{\partial T}{\partial \bar{y}} + \bar{u} \frac{\partial T}{\partial \bar{x}} = k_{nf} \frac{\partial^2 T}{\partial \bar{y}^2} \quad (3)$$

The flow problem is analyzed under the following important boundary conditions to guarantee convergence and uniqueness of the solution for each unknown.

$$\begin{aligned} \text{At } \bar{y}^- = 0 : \bar{u}(\bar{x}, \bar{y}^-) &= \bar{u}_w(\bar{x}) = b\bar{x}, \quad \bar{v} = 0, \quad T(\bar{x}, \bar{y}^-) = T_w \\ \text{As } \bar{y}^- \rightarrow \infty : \bar{u}(\bar{x}, \bar{y}^-) &\rightarrow \bar{u}_e(\bar{x}) = a\bar{x}, \quad T(\bar{x}, \bar{y}^-) \rightarrow T_\infty \end{aligned} \quad (4)$$

£.I. Thermophysical attributes

The thermophysical characteristics of the base fluid and nanoparticles are detailed in Table 2, while Table 2 outlines the characteristics of the resulting nanofluid. In these tables, subscripts are used to distinguish between different components: *f* denotes the base fluid, *s* represents the solid nanoparticles, and *nf* refers to the nanofluid.

Table 1: Key thermophysical properties of nanofluids

Properties	Nanofluid
Thermal expansion coefficient β_f	$\rho_{nf}\beta_{nf} = \beta_f\rho_f + \rho_s\beta_s - \rho_f\beta_f \phi$
Heat capacity $(\rho C_p)_f$	$(\rho C_p)_{nf} = \rho_f(C_p)_f + \rho_s(C_p)_s - \rho_f(C_p)_f \phi$
Density ρ_f	$\rho_{nf} = \phi\rho_s + (1 - \phi)\rho_f$
viscosity μ_f	$\mu_{nf} = \mu_f(1 - \phi)^{-2.5}$
Thermal conductivity k_f	$k_{nf} = \frac{k_s(2\phi + 1) - 2k_f(\phi - 1)}{(\phi + 2)k_f - (\phi - 1)k_s} k_f$

Table 2: Thermophysical property data of blood and copper (Cu) nanoparticle in numerical form [20, 21].

Properties	Cu	Blood
$C_p \text{ (J kg}^{-1} \text{ K}^{-1}\text{)}$	385	3594
$\rho \text{ (kg m}^{-3}\text{)}$	8933	1063
$\beta \text{ (10}^{-6}\text{K}^{-1}\text{)}$	1.67	0.18
$\mu \text{ (kg m}^{-1}\text{s}^{-1}\text{)}$	-	1
$k \text{ (Wm}^{-1} \text{ K}^{-1}\text{)}$	401	0.492

£.£. Non-dimensionalization

Dimensionless variables are introduced to convert the equation system (1) -(3) into their dimensionless counterparts [22]:

$$\frac{\bar{u}}{f} = u, \quad \bar{v} = v, \quad \frac{T - T_\infty}{T_w - T_\infty} = \theta, \quad \frac{\bar{x}}{a} = x, \quad \frac{\bar{y}^-}{\sqrt{a\nu}} = y$$

$$, v = \sqrt{a_v} \quad , \vartheta = \frac{T_{\psi} - T_{\infty}}{T_{\psi} - T_{\infty}} \quad , x = , y = q_v \cdot \frac{f}{a} \quad q_v \quad (5)$$

Equations (1) to (4), encompassing the continuity equation and boundary conditions, have been transformed into their non-dimensional forms as:

$$\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0, \quad (6)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = x + \frac{1}{A} \frac{1}{1+\beta} \frac{\partial^2 u}{\partial y^2} + \frac{B\beta_f}{a\sqrt{\alpha_f}} (T - T_\infty) g, \quad (7)$$

$$v \frac{\partial T}{\partial y} + u \frac{\partial T}{\partial x} = \frac{1}{C} \frac{k_{nf}}{k_f} \frac{1}{Pr} \frac{\partial^2 T}{\partial y^2}, \quad (8)$$

with required boundary conditions (BCs)

$$\begin{aligned} \text{At } y = 0: \quad & u(x, y) = \lambda x, \quad v(x, y) = 0, \quad T(x, y) = T_w, \\ \text{As } y \rightarrow \infty: \quad & u(x, y) \rightarrow x, \quad T(x, y) \rightarrow T_\infty. \end{aligned} \quad (9)$$

The λ , stretching/shrinking parameter is defined as b/a . The Prandtl number Pr , denoted as $= \frac{\nu_f}{\alpha_f}$. Additionally, expressions for constants A and B as functions of ϕ can be derived as follows:

$$A = (1-\phi)^{2.5} \frac{\rho_s}{1-\phi + \phi \frac{\rho_s}{\rho_f}}, \quad B = \frac{1-\phi + \phi \frac{(\rho\beta)_s}{(\rho\beta)_f}}{1-\phi + \phi \frac{\rho_s}{\rho_f}}, \quad C = \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} - \phi + 1$$

In two-dimensional incompressible flow, the stream function (denoted as ψ) is a scalar function that simplifies the analysis of fluid motion. It is defined such that the velocity components can be expressed as:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

These definitions inherently satisfy the continuity equation for incompressible flow, ensuring mass conservation without directly solving it. Utilizing the stream function allows to rewrite the governing equations in non-dimensional form, facilitating the analysis of fluid behavior under various conditions.

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \psi}{\partial x} = x + \frac{1}{A} \frac{1}{1+\beta} \frac{\partial^3 \psi}{\partial y^3} + \frac{B\beta_f}{a\sqrt{\alpha_f}} g(T_w - T_\infty) \vartheta, \quad (10)$$

$$\frac{\partial \vartheta}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \vartheta}{\partial y} \frac{\partial \psi}{\partial x} = \frac{1}{C} \frac{k_{nf}}{k_f} \frac{1}{Pr} \frac{\partial^2 \vartheta}{\partial y^2}, \quad (11)$$

The associated BCs are:

$$\begin{aligned} \frac{\partial \psi}{\partial y} = \lambda x, \quad \frac{\partial \psi}{\partial x} = 0, \quad \vartheta = 1, \quad \text{at } y = 0, \\ \frac{\partial \psi}{\partial y} \rightarrow x, \quad \vartheta \rightarrow 0, \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (12)$$

£. #. Group of scaling transformations

Lie group methods are powerful tools for finding solutions to differential equations that remain unchanged under certain transformations [23]. These methods systematically identify possible similarity transformations, which are changes of variables that simplify complex equations. By applying these transformations, we can simplify it by reducing independent variables, turning complex PDEs into simpler ODEs. This transformation retains the invariance of the system along with its boundary conditions. The resulting similarity variables are defined as follows:

$$\Gamma : \quad x^* = x e^{\varepsilon \xi_1}, \quad y^* = y e^{\varepsilon \xi_2}, \quad \psi^* = \psi e^{\varepsilon \xi_3}, \\ \vartheta^* = \vartheta e^{\varepsilon \xi_4}, \quad \theta_f^* = \theta_f e^{\varepsilon \xi_5}. \quad (13)$$

In this context, the parameter $\varepsilon = 0$ stand in for the group parameter, while ξ_i ($1 \leq i \leq 5$) are any real numbers serving as transformation parameters. It's important to note that not all these parameters are zero simultaneously. Equation (13) illustrates the transformations applied to $(x, y, \psi, \vartheta, \theta_f)$, mapping them to the coordinates $(x^*, y^*, \psi^*, \vartheta^*, \theta_f^*)$. Using Eq. (13), then Eqs. (10) and (11) become

$$e^{\varepsilon(\xi_1 + 2\xi_2 - 2\xi_3)} \frac{\partial \psi^*}{\partial y^*} \frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial^2 \psi^*}{\partial y^{*2}} = x^* e^{-\varepsilon \xi_1} + \frac{e^{\varepsilon(3\xi_2 - \xi_3)}}{A} \left(1 + \frac{1}{\theta} \frac{\partial^3 \psi^*}{\partial y^{*3}} \right) \\ + e^{-\varepsilon(\xi_4 + \xi_5)} \frac{B \theta_f^*}{a \sqrt{\frac{f}{\alpha \nu_f}}} g(T_w - T_\infty) \vartheta^*, \\ e^{\varepsilon(\xi_1 + \xi_2 - \xi_3 - \xi_4)} \frac{\partial \psi^*}{\partial y^*} \frac{\partial \vartheta^*}{\partial x^*} - \frac{\partial \vartheta^*}{\partial y^*} \frac{\partial \psi^*}{\partial x^*} = \frac{e^{\varepsilon(2\xi_2 - \xi_4)} k_{nf}}{Pr} \frac{1}{k_f B} \frac{\partial^2 \vartheta^*}{\partial y^{*2}},$$

and the related BCs are:

$$\text{At } y^* = 0 : e^{\varepsilon(\xi_2 - \xi_3)} \frac{\partial \psi^*}{\partial y^*} = e^{-\varepsilon \xi_1} x^* \lambda, \quad e^{\varepsilon(\xi_1 - \xi_3)} \frac{\partial \psi^*}{\partial x^*} = 0, \quad e^{-\varepsilon \xi_4} \vartheta^* = 1, \\ \text{As } y^* \rightarrow \infty : e^{\varepsilon(\xi_2 - \xi_3)} \frac{\partial \psi^*}{\partial y^*} \rightarrow e^{-\varepsilon \xi_1} x^*, \quad e^{-\varepsilon \xi_4} \vartheta^* \rightarrow 0.$$

Ensuring the invariance of the system with respect to the applied transformation group, the corresponding relationships between the parameters are listed as follows:

$$\xi_1 + 2\xi_2 - 2\xi_3 = 3\xi_2 - \xi_3 = -\xi_1 = -\xi_4 - \xi_5; \\ \xi_1 + \xi_2 - \xi_3 - \xi_4 = 2\xi_2 - \xi_4; \\ -\xi_1 = \xi_2 - \xi_3; \\ -\xi_4 = 0. \quad (14)$$

The relations presented in Eq.(14) offer the solutions:

$$\xi_4 = 0 = \xi_2 \quad \text{and} \quad \xi_1 = \xi_3 = \xi_5.$$

The transformations presented in Eq. (13) can be further simplified by utilizing the

relationships among the variables ξ .

$$x^* = x e^{\varepsilon \xi_1}, \quad y^* = y, \quad \psi^* = \psi e^{\varepsilon \xi_1}, \quad \vartheta^* = \vartheta, \quad \theta_f^* = \theta_f e^{\varepsilon \xi_1}. \quad (15)$$

By applying a Taylor series expansion to Eq. (15) around $\varepsilon = 0$ and retaining terms up to the first order in ε , we derive the following characteristic equations:

$$\frac{dx}{\xi_1 x} = \frac{dy}{0} = \frac{d\vartheta}{0} = \frac{d\psi}{\xi_1 \psi} = \frac{d\theta_f}{\xi_1 \theta_f}.$$

The following similarity transformations arise from the scaling group transformations:

$$\vartheta = \vartheta(\eta), \quad y = \eta, \quad \psi = x f(\eta), \quad \theta_f = x \theta_{fo}. \quad (16)$$

In this context, the constant θ_{fo} represents the thermal expansion coefficient.

Substituting the expression from Eq. (16) into Eq.(10) to (12) yields:

$$1 + \frac{1}{\beta} f''' + A f'' f - A f'^2 + A B \Lambda \vartheta + 1 = 0, \quad (17)$$

$$\frac{1}{Pr} \frac{k_{nf}}{k_f} \frac{1}{C} \vartheta'' + f \vartheta' = 0, \quad (18)$$

We define the boundary conditions as:

$$\begin{aligned} f'(0) &= \lambda, \quad f(0) = 0, \quad \vartheta(0) = 1, \\ \text{As } \eta \rightarrow \infty : f'(\eta) &\rightarrow 1, \quad \vartheta(\eta) \rightarrow 0. \end{aligned} \quad (19)$$

A crucial factor is the modified mixed convection parameter, represented by the parameter Λ , which is defined as follows:

$$\Lambda = \frac{\theta_{fo} g (T_w - T_\infty)}{a \sqrt{av_f}}.$$

£.J. Physical quantities

The effectiveness of numerous industrial and technological processes involving fluid flow is largely governed by the fluid transport rates close to the solid—fluid boundary. Among these important parameters is the local Nusselt number and wall friction coefficient, given as:

The surface friction coefficient is a dimensionless quantity that characterizes the resistance to fluid flow due to viscosity at the surface. It is expressed as:

$$C_f = \frac{2 \tau_w^-}{\rho_f \bar{u}^2_e}$$

where $\tau_w^- = \mu_{nf} \left(1 + \frac{1}{\beta} \frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0}$ is the wall shear stress.

The local Nusselt number quantifies the enhancement of heat transfer due to con-

vection relative to conduction. It is defined as:

$$Nu_{x^-} = \frac{\bar{q}_w \bar{x}}{k_f (T_w - T_\infty)}$$

$$\text{where the wall heat flux } \bar{q}_w = -k_{nf} \left. \frac{\partial T}{\partial y} \right|_{y^-=0}$$

The dimensionless quantities defined in Eqs. (5) and (16) are employed to derive the subsequent results.

$$\sqrt{Re_x} C_f = 2 \frac{\mu_{nf}}{\mu_f} \left(1 + \frac{1}{\theta} f''(0) \right) \quad \text{and} \quad \frac{Nu_x}{Re_x} = - \frac{k_{nf}}{k_f} \theta'(0). \quad (20)$$

In the context of fluid dynamics, the local Reynolds number that characterizes the flow behavior at a specific point within the fluid. It is defined as $Re_{x^-} = \frac{\bar{u}_w(\bar{x}) \bar{x}}{\nu_f}$.

3. Numerical approach

The higher-order non-dimensional ODEs (17) and (18), together with their BCs (19), are first converted into a system of first-order differential equations to form a boundary value problem. These equations are then numerically solved using MATLAB's built-in bvp4c solver, which is well-suited for handling such boundary value problems. Compared to other approaches, bvp4c offers greater reliability, provided an initial guess that satisfies the boundary conditions is supplied. With a stringent convergence criterion of 10^{-10} , the bvp4c solver demonstrates exceptional performance in solving non-dimensional nonlinear ODEs.

bvp4c utilizes a collocation approach based on the three-stage Lobatto IIIa formula, providing a fourth-order accurate solution that is continuously differentiable across the interval. The solver automatically adjusts the computational mesh to enhance accuracy, making it effective for stiff or complex systems. Users can specify parameters like error tolerances and initial guesses using the “bvpinit” function, allowing the solver to handle various scenarios with high precision.

Let

$$f = z_1, \quad f' = z'_1 = z_2, \quad f'' = z'_2 = z_3, \quad \vartheta = z_4, \quad \vartheta' = z'_4 = z_5.$$

Equations (17) and (19) have been transformed into a system of first-order differential equations to facilitate numerical analysis.

$$1 + \frac{1}{\theta} z'_3 + A z_1 z_3 - A z_2^2 + A B \Lambda z_4 + 1 = 0, \quad (21)$$

$$\frac{1}{Pr} \frac{k_{nf}}{k_f} \frac{1}{C} z'_5 + f z_5 = 0, \quad (22)$$

with reduced boundary follow as:

$$\begin{aligned} z_2(0) &= \lambda, \quad z_1(0) = 0, \quad z_4(0) = 1, \\ z_2(\eta) &= 1, \quad z_4(\eta) = 0, \quad \text{as } \eta \rightarrow \infty. \end{aligned} \quad (23)$$

#.I. Validation

Table 3 presents a comparative analysis of the skin friction coefficients determined for varying Pr . The results align closely with those reported by Wang, and Gorla and Sidawi, indicating consistency and reliability in the findings.

Table 3: Analyzing the values of $-\theta'(0)$ under the conditions where ϕ is zero and $\theta \rightarrow \infty$

Pr	Present	Wang [24]	Gorla and Sidawi [25]
2	0.91141	0.9114	0.9114
7	1.89544	1.8954	1.8905
20	3.35392	3.3539	3.3539

4. Outcomes and discussions

The graphical analysis of the nanofluid flow reveals how variations in key parameters influence the dimensionless velocity and temperature profiles. Specifically, increasing the Casson parameter (θ) and the modified mixed convection parameter (Λ) tends to decrease the velocity profile while enhancing the temperature distribution. These effects are attributed to the non-Newtonian characteristics of the Casson fluid, where higher θ values increase fluid viscosity, leading to reduced flow velocity and elevated thermal boundary layers. Furthermore, the study examines the impact on the Nusselt number and surface friction. An increase in θ and Λ results in a higher heat transference, indicating improved heat transfer efficiency, while surface friction decreases, suggesting reduced shear stress at the boundary.

For the simulations, unless specified otherwise, the parameters are set as follows: $Pr = 21$, $\phi = 0.01$, $\Lambda = 0.5$, and $\theta = 1.2$ [26, 27]. These settings provide a baseline for analyzing the influence of parameter variations on the flow and thermal characteristics of the nanofluid.

J.I. Results based on velocity and temperature graphs:

The results from the velocity and temperature graphs show that increasing the Casson parameter leads to an increase in fluid velocity (Fig. 2(b)) because a higher Casson parameter represents reduced yield stress in the fluid, meaning the copper-infused blood becomes less resistant to flow. As a result, the fluid moves more freely. On the other hand, a decrease in temperature can be observed in Fig. 2(a) for both stretching and shrinking sheet cases because with reduced internal resistance due to a higher Casson parameter, the fluid generates less internal heat through friction. This leads to more efficient heat transfer and a lower fluid temperature (which is analogous to Hayat et al. [28]). This behavior highlights the potential of Casson fluids in applications such as biomedical devices and targeted thermal therapies.

The study reveals that increasing the volume fraction of copper nanoparticles in the blood (Casson fluid) enhances the fluid's velocity (Fig. 3(b)) and reduces its temperature (Fig. 3(a)) in both stretching and shrinking sheet scenarios. Physically, the rise in velocity is due to improved thermal conductivity and momentum diffusion from the copper nanopar-

ticles, which assist the fluid flow. Meanwhile, the decrease in temperature occurs because

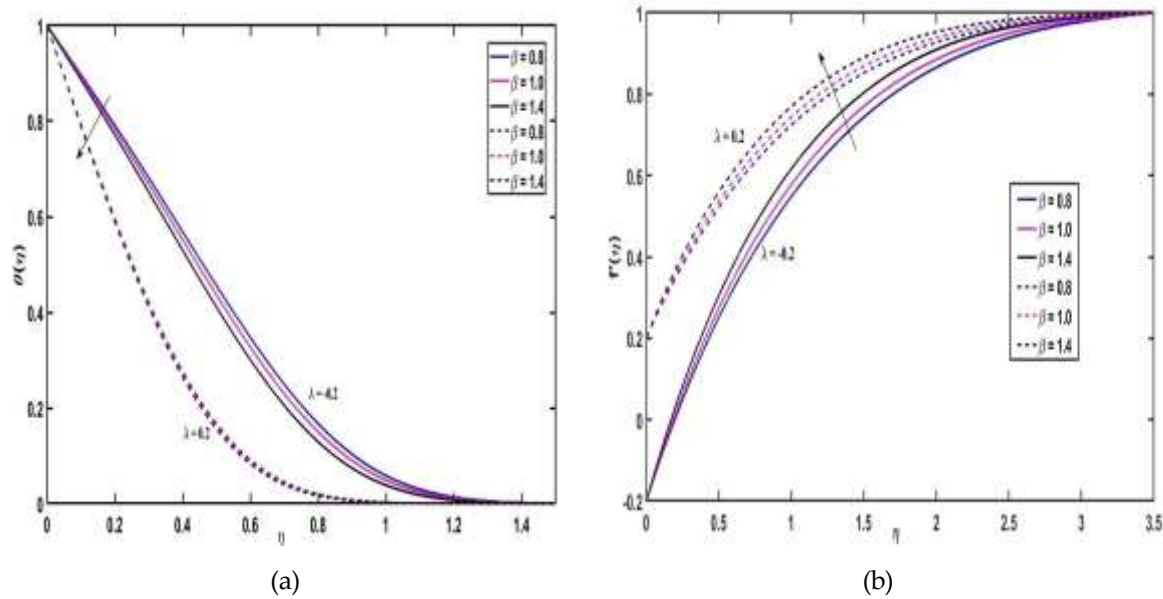


Figure 2: Influence of β on (a) Temperature and (b) Velocity profiles

the high thermal conductivity of copper facilitates efficient heat dissipation from the fluid to the surrounding environment, leading to a cooler fluid profile. This behavior is beneficial in biomedical and industrial applications where improved cooling and enhanced flow are desired. An increase in the modified mixed convection parameter leads to a higher fluid velocity

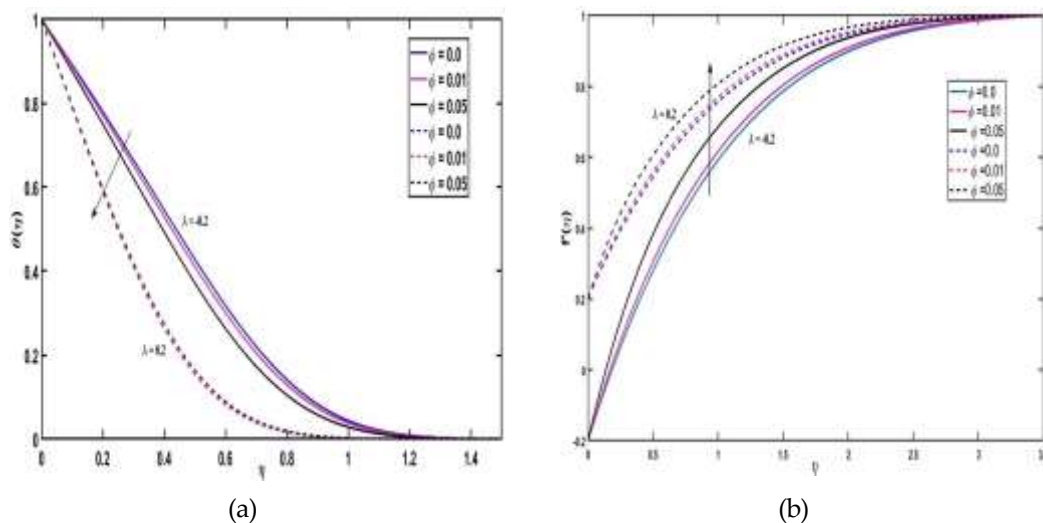


Figure 3: Variations in (a) Temperature and (b) Velocity profiles with varying ϕ

(Fig. 4(b)). This is because the combined effect of buoyancy and the movement of sheet enhances the overall momentum of the fluid, resulting in a thinner momentum boundary layer and accelerated flow. On the other side, the fluid temperature decreases (Fig. 4(a))

because the augmented convective motion facilitates more efficient heat transfer away from

the fluid, leading to a thinner thermal boundary layer and a cooler fluid profile.

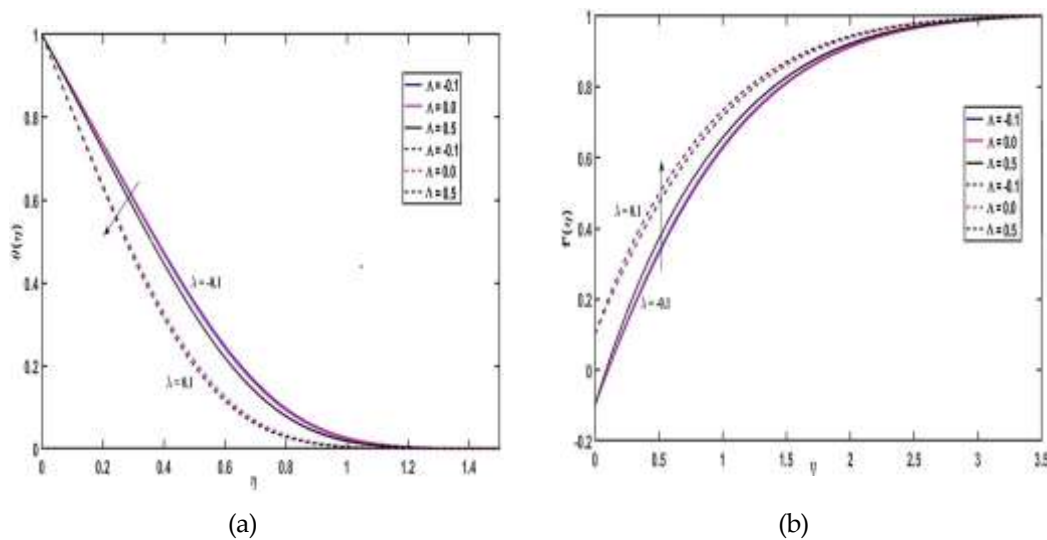


Figure 4: Influence of Λ on (a) Temperature and (b) Velocity profiles

J.£. Effects on surface friction and heat transfer rate:

A higher Casson parameter indicates stronger non-Newtonian behavior, reducing velocity gradients at the surface and consequently lowering viscous drag, which is shown in Fig. 5(a). The skin friction is lower for stretching sheets. The stretching motion helps stabilize the boundary layer, reducing flow separation and resulting in less resistance (lower shear stress) at the surface. A higher Casson parameter reduces the fluid's viscosity, enhancing thermal conductivity. This improvement in heat transfer efficiency leads to an increased Nusselt number (see Fig. 5(b)), indicating a higher rate of heat transfer from the surface. The Nusselt number is higher for stretching sheets. The stretching action maintains a more stable thermal boundary layer, enhancing the transfer of heat from the sheet to the fluid.

In Fig. 6(a), an increase in the mixed convection parameter leads to a decrease in skin friction. This is because the enhanced buoyancy forces assist the fluid motion, reducing the shear stress at the surface. With higher mixed convection, the convective heat transfer is augmented, resulting in a higher Nusselt number. This indicates more efficient heat removal from the surface due to the combined effects of natural and forced convection. This configuration exhibits lower skin friction and higher Nusselt numbers in Fig. 6(b). The diverging nature of the flow in stretching sheets reduces shear stress at the surface, leading to lower skin friction. Simultaneously, the thermal boundary layer becomes thinner, increasing the thermal gradients at the boundary, which contribute to elevate heat transfer rates.

As the Prandtl number increases in Fig. 7(a), the friction coefficient decreases. This indicates a reduction in shear stress at the surface due to lower momentum diffusivity. The stretching action reduces the velocity gradient at the wall, decreasing shear stress. Simultaneously, the Nusselt number increases, signifying enhanced heat transfer rates (see Fig. 7(b)). This is because higher Prandtl numbers correspond to lower thermal diffusivity, leading to

steeper temperature gradients near the surface.

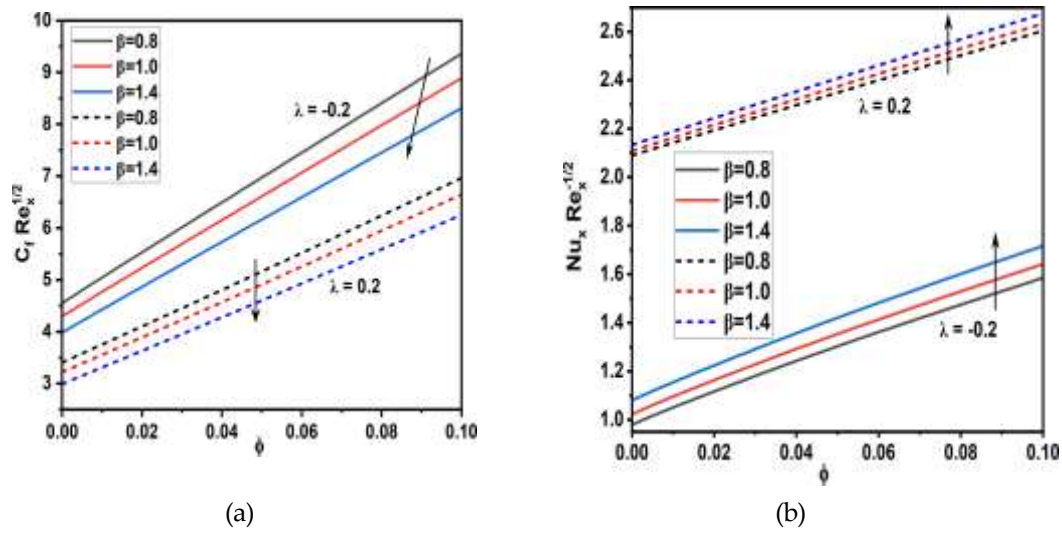


Figure 5: Variations of β on (a) Skin friction and (b) Nusselt number with varying ϕ

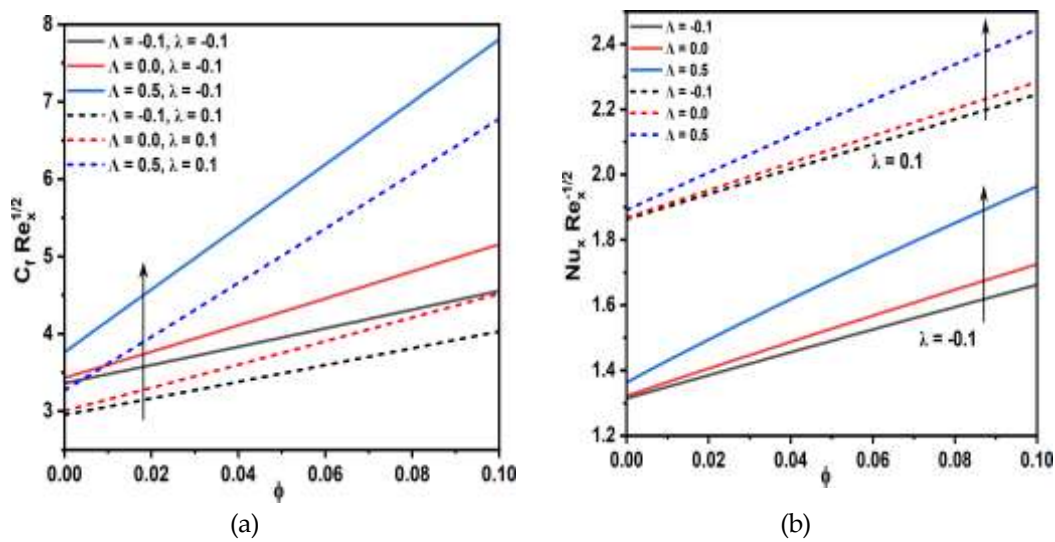


Figure 6: Variations of Λ on (a) Skin friction and (b) Nusselt number with varying ϕ

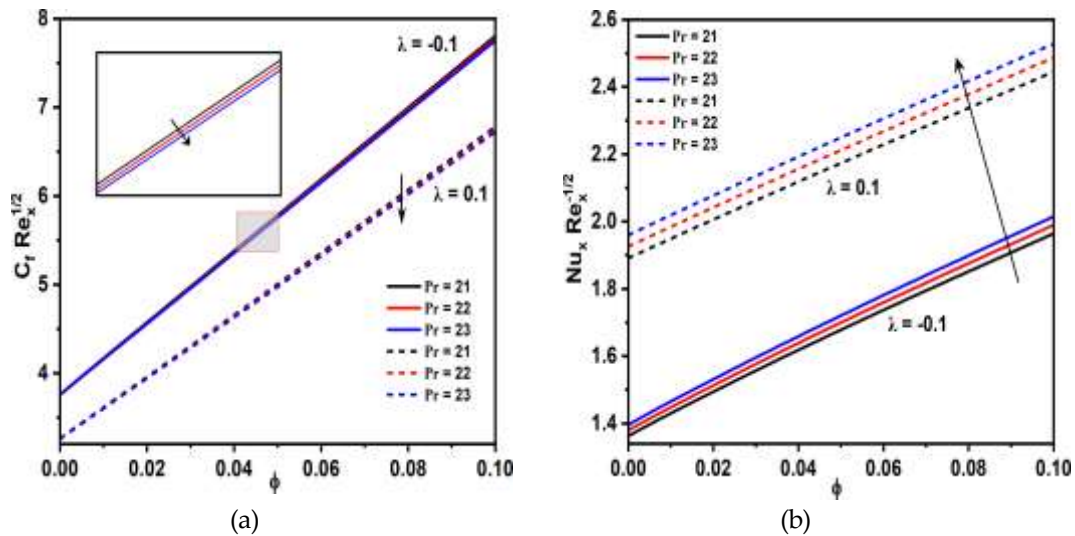


Figure 7: Variations of Pr on (a) Surface friction and (b) Nusselt number with varying ϕ

5. Concluding remarks

This research presents a numerical analysis of Casson nanofluid flow, specifically focusing on blood infused with copper nanoparticles, over a stretchable sheet under the influence of natural convection. Similarity transformations simplify the model equations from PDEs into ODEs, which are then handled computationally with MATLAB's `bvp4c` solver. The study examines the impacts of various parameters in flow regimes and heat transfer efficiency, with results validated against existing literature to ensure accuracy. Key findings are illustrated through graphical representations, highlighting the impact of different physical parameters on the nanofluid behavior:

Elevating the volume fraction of copper nanoparticles results in higher friction and heat transfer rate.

In the stretching sheet scenario, the fluid experiences lower skin friction and higher Nusselt numbers compared to the shrinking sheet case.

Increasing the Casson parameter reduces the fluid's velocity. Simultaneously, it elevates the fluid temperature by thickening the thermal boundary layer.

Increasing the Prandtl number reduces skin friction and enhances heat transfer efficiency making it beneficial for applications like electronic cooling systems where efficient thermal management is crucial.

These findings provide valuable insights for optimizing thermal and flow characteristics in biomedical and engineering applications, like precision drug delivery technologies and cooling technologies, where precise control over heat and momentum transfer is crucial.

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