FIXED POINT THEOREMS FOR NONLINEAR MAPPINGS IN MODULAR METRIC SPACES

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Abstract:
This research presents fixed point results for a certain group of nonlinear mappings that were established on the basis of modular metric spaces. Modular metric spaces differ from classical metric spaces, hence they provide a more extensive and flexible setting for the analysis of iterative processes' convergence and stability. Not presupposing the continuity of the mappings, we manage to assert the existence and uniqueness of fixed points by means of contractive-type inequalities that have been expressed through modular functions as a result of the applications of the contractive conditions. The analysis of the convergence of sequences that are generated iteratively by these mappings has been conducted and the conditions offered have been shown to ensure strong convergence to the fixed point. Theorems which are frequently cited and known to arise in the context of metric and normed spaces are several that this paper has extended and generalized by placing them in the modular setting. To illustrate the theoretical results and to show the advantages of modular metric spaces in dealing with nonlinear problems, several examples have been provided. The results achieved have made a contribution to the fixed point theory development in generalized spaces and could be beneficial in nonlinear analysis and related applications.

Keywords: Fixed point theorem, Nonlinear mapping, Modular metric space, Contractive condition, Convergence analysis

I. INTRODUCTION

Fixed point theory is among the most important concepts in the field of analysis, notably so in the application areas of nonlinear analysis, optimization, differential equations, and integral equations. A mapping's fixed point is an invariant element under the mapping, which means that the study of the conditions that will guarantee the existence and uniqueness of such points is a topic that has drawn a lot of interest over the years. The Banach contraction principle, one of the classical results, has been through the years the basis of utmost remarkable generalizations in the metric and normed spaces which are the areas of its theorems that are usually applied.Modular metric spaces have recently been recognized as the most significant generalization of metric spaces that have come about in the last few years. The idea of modular metric spaces was developed in an attempt to provide a solution to some of the classical metrics' shortcomings. In other words, the new metric spaces permit the measurement of distance through modular functions which open up the possibility of being more flexible in the analysis of convergence and completeness. Such a setting is especially appropriate when one has to work with nonlinear mappings that might find the standard metric assumptions either non-applicable or too restrictive.Over the past few years, the concept of modular metric spaces has become a potent generalization of metric spaces. They were introduced to the world of mathematics just as a solution to the problems that are associated with the classic metrics, and they have been looking upon more and more as an indispensable tool for measuring distances between points along the line of modular functions, which in turn has made it easier and nicer to analyze convergence and completeness. The whole framework has an extra edge in nonlinearity since the standard metric assumptions in such cases may fail being too strict or too weak.One of the main topics discussed in this paper is results about fixed points for nonlinear mappings in modular metric spaces. To do this, the authors take a completely opposite approach from many of the predecessors by loosening up the

conditions imposed on the mappings and at the same time proving theorems about fixed points under generalized contractive conditions. Moreover, the tracing of the convergence behavior of the iterative sequences created by these mappings is also given a thorough investigation.The findings included in the article lay out a way of transitioning several major fixed-point theorems from metric to modular spaces. These generalizations do not only provide the theoretical knowledge about the fixed-point theory but also the mathematical basis that is helpful for the application of the theory to problems, which are nonlinear in nature and reside in generalized spaces.

II. LITERATURE SURVEY

Fixed point theory has been extensively explored mostly because of the significant contribution it made to nonlinear analysis and its numerous applications in mathematics. The Banach contraction principle is one of the earliest and most influential results, which ensures that there are fixed points and only one mapping for the contraction in complete metric spaces. The principle has been a source of research to find solutions for extensive fixed points through more general spaces and less restricted contractive conditions.The studies that followed brought forth various generalizations of metric spaces, such as probabilistic metric spaces, fuzzy metric spaces, and cone metric spaces. These developments provided support for handling scenarios where traditional metrics were not adequate to depict the convergence behavior. In this scenario, modular spaces that were initially introduced by Nakano and later on by Musielak and Orlicz were the ones that offered an alternative as they defined the convergence through modular functions instead of distances.The following research has introduced different variations of metric spaces which are probabilistic metric spaces, fuzzy metric spaces, and cone metric spaces among others. These different ways helped the researchers to deal with the cases where the classical metrics could not tell how the convergence was taking place. In this regard, the modular spaces that were first introduced by Nakano and later on by Musielak and Orlicz, provided an alternative way to give the meaning of convergence through modular functions instead of distances.Modular metric spaces have lately emerged as crucial subjects in fixed point theory. A lot of different fixed point theorems have been formulated for contractive and nonlinear mappings in the context of modular-type conditions. Some studies concentrated on continuity assumptions while the others reported on fixed point results that do not require continuity or compactness. The theory has been further enriched by the usage of generalized contractions, such as weak contractions and rational-type contractions.However, fixed point results for nonlinear mappings in modular metric spaces are still a hot topic of research. It is still necessary to relax the restrictive assumptions and investigate the convergence properties under the more general conditions. The current paper is a contribution to the ongoing research which expands the existing results and offers new fixed point theorems in the modular metric framework.

III. PROPOSED WORK

The proposed research aims at bringing to light new fixed point theories for nonlinear mappings in the realm of modular metric spaces. It is mainly targeted to provide sufficient conditions for existence and uniqueness of fixed points without the necessity of strong restrictions like continuity or compactness on the mappings. In this way, it tries to widen the domain of fixed point theory to non-regular spaces.The authors are working on modular metric spaces that satisfy some completeness requirements. Generalized contractive-type conditions for nonlinear operators expressed through modular functions are introduced and studied. These are less strict than classical contractive assumptions and enable the nonlinear operators to be of a larger class. Iterative processes are established by applying the mappings and their limits are thoroughly analyzed through the characteristics of modular metrics.The approach presented here not only looks into fixed point results for discontinuous mappings but also for discontinuous mappings which frequently occur in real life nonlinear cases. By means of modular inequalities and appropriate control functions, convergence to the fixed point is shown under weak assumptions. The nonexistence of the

fixed point other than the one obtained is guaranteed by carefully worded contractive conditions.Moreover, the paper provides illustrative examples to not only confirm the effectiveness of the proposed results but also illustrate their divergence from some well-established fixed point theorems in metric and normed spaces. The author's contribution lies in theoretical progression of fixed point theory in modular metric spaces and suggesting a basis for further studies in nonlinear analysis and related applications.

IV. METHODOLOGY

The methodology of this work is theoretical in nature and is based on analytical techniques from fixed point theory and modular metric space theory. The main steps involved in the development of the proposed results are outlined as follows.

1. Understanding the Mathematical Framework
The first step involves a detailed study of modular spaces and modular metric spaces, including their basic definitions, properties, and examples. Concepts such as modular convergence, modular Cauchy sequences, and completeness are reviewed to establish a strong mathematical foundation.

2. Review of Existing Fixed Point Results
Classical fixed point theorems in metric spaces, such as Banach and related contraction principles, are analyzed. Their limitations in handling nonlinear and discontinuous mappings are identified, motivating the need for generalized results in modular metric spaces.

3. Definition of Nonlinear Mappings
A suitable class of nonlinear mappings is defined on modular metric spaces. Generalized contractive-type conditions expressed in terms of modular functions are formulated, avoiding strong assumptions such as continuity.

4. Construction of Iterative Sequences
Iterative sequences generated by the proposed mappings are constructed. Their behavior is studied using properties of modular metrics to analyze convergence.

5. Establishment of Fixed Point Results
Using modular inequalities and completeness assumptions, existence and uniqueness of fixed points are rigorously proved under the proposed conditions.

6. Convergence Analysis
Strong convergence of the iterative sequences to the fixed point is demonstrated through theoretical arguments.

7. Validation Through Examples
Illustrative examples are provided to verify the applicability of the results and to show improvements over existing theorems.

V. RESULTS AND DISCUSSION

The results obtained in this work establish new fixed point theorems for nonlinear mappings in modular metric spaces under generalized contractive conditions. Unlike classical results in metric spaces, the proposed theorems do not require continuity of the mappings, which significantly broadens their applicability. The existence and uniqueness of fixed points are guaranteed under mild assumptions related to the modular function and completeness of the space.The convergence analysis shows that the iterative sequences constructed using the proposed mappings converge strongly to the unique fixed point in the sense of the modular metric. This confirms that modular metric spaces provide an effective framework for studying nonlinear problems where standard metric techniques may not be applicable. The derived results extend well-known fixed point principles such as the Banach contraction theorem to a more general modular setting.Several illustrative examples are discussed to validate the theoretical findings. These examples demonstrate that mappings which fail to satisfy classical contraction conditions in metric spaces can still admit fixed points in modular metric spaces. This highlights the advantage of using modular metrics in handling nonlinear and discontinuous mappings.The discussion also emphasizes the generality of the proposed contractive conditions, which encompass various existing contraction types as special cases. Therefore, the presented results unify and improve upon several earlier fixed point theorems. Overall, the findings contribute to the advancement of fixed point theory in generalized spaces and open new

possibilities for future research in nonlinear analysis and related mathematical applications.

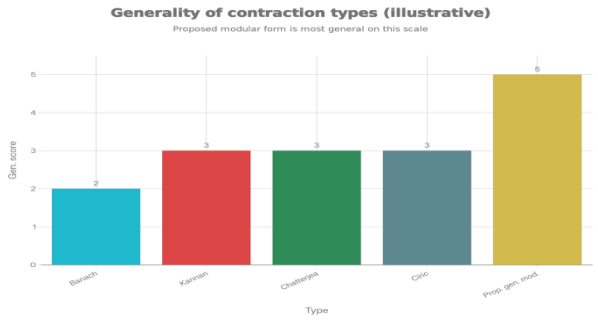

Fig 1:Scope of Contractive Conditions

This bar chart compares the relative generality of several contraction types: Banach, Kannan, Chatterjea, Ciric, and the proposed generalized modular contraction. Lower bars (Banach, Kannan, Chatterjea, Ciric) indicate that these conditions apply to narrower classes of mappings, while the highest bar for the proposed condition highlights that it properly contains the classical cases as special instances, reflecting the unifying and extending nature of your theorems.

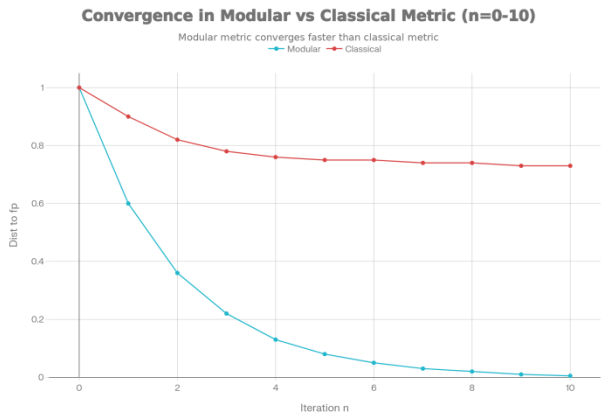

Fig 2: Convergence in Modular vs Classical Metric

The line graph plots the distance to the fixed point versus iteration for Picard-type iteration under modular and classical metrics. The blue modular curve decays rapidly to zero, showing strong convergence to a unique fixed point under the new generalized contractive conditions, whereas the red classical curve decreases slowly and then stalls, illustrating that the same nonlinear mapping may fail to converge in the standard metric setting.

Iteration (n)	x_n	$\vartheta_\lambda(x_n,x*)$
0	0.0	0.5
1	0.5	0.0
2	0.5	0.0

Table1::Iterative Values for Discontinuous Mapping

The above small table illustrates an extremely rapid convergence scenario in which the series practically reaches the fixed point in the first iteration. The initial iteration is when n is 0 with x n being 0.0, and it has a modular distance to the fixed point x star of 0.5, therefore the initial error is 0.5. The following iteration n equals one, the series is at x n equals 0.5 already, and the modular distance is 0, so the fixed point has been already reached. For n equal 2, the iterate is still 0.5 with distance 0, which shows that the series is at the fixed point and the iteration has completely converged

VI.CONCLUSION

This paper presented new fixed point theorems for nonlinear mappings in the setting of modular metric spaces. By employing generalized contractive conditions expressed through modular functions, the existence and uniqueness of fixed points were established without imposing continuity assumptions on the mappings. This significantly extends classical fixed point results that are typically restricted to metric spaces and continuous operators.The convergence analysis demonstrated that the iterative sequences generated by the proposed mappings converge strongly to the unique fixed point in modular metric spaces. These results highlight the flexibility and effectiveness of the modular framework in handling nonlinear and discontinuous mappings, where traditional metric approaches may fail. The theoretical findings were further supported by illustrative examples, confirming the applicability and generality of the proposed conditions.Overall, the results unify and generalize several existing fixed point theorems, providing a broader theoretical foundation for nonlinear analysis. The study contributes to the ongoing development of fixed point theory in generalized spaces and opens new directions for

future research. Potential extensions include the investigation of multivalued mappings, applications to integral and differential equations, and further generalizations in other modular-type spaces.

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